USP

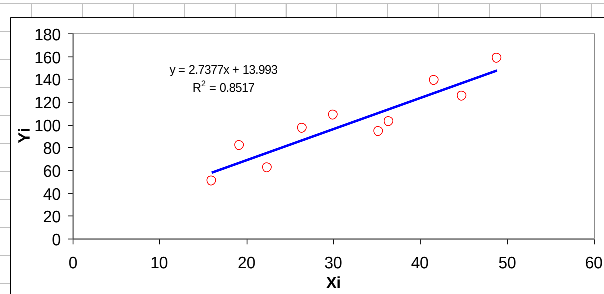
Biostatistics II

Model parameter estimations: Confronting models with measurements

Fitting models to data

- We all know how to do this

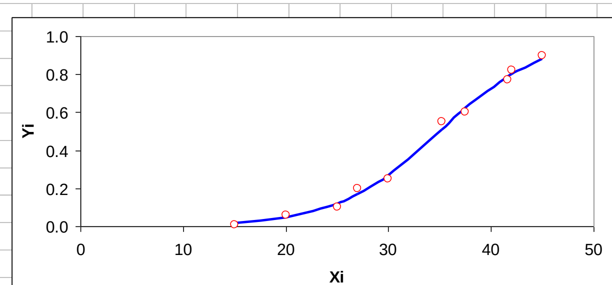
i	Xi	Yi
1	16.0	50.79
2	19.2	81.46
3	22.4	62.26
4	26.4	96.57
5	30.0	108.28
6	35.2	94.05
7	36.4	102.85
8	41.6	138.61
9	44.8	124.75
10	48.8	158.57



$$Y = a + bX$$

- But do we all know how to do this?

i	Xi	Yi
1	15.0	0.02
2	20.0	0.05
3	25.0	0.12
4	27.0	0.17
5	30.0	0.27
6	35.2	0.51
7	37.5	0.62
8	41.6	0.79
9	42.0	0.80
10	45.0	0.88



$$Y = \frac{1}{1 + e^{-k\ X - b}}$$

Overview

- Objectives:
  - Introduce the basic concept and the basic procedure for fitting a mathematical model to a set of measurements
  - The method for obtaining the parameter estimates (“best fit”) is the same irrespective of the model complexity
    - The small print: Applies only when the error model is the same
- Outline
  - Goodness of fit: The concept of minimum sums of squares
  - The concept of minimization explained with linear regression examples
  - Model violation

Model fitting procedure

- Three essential requirements:
  - Observations from a population
  - A formal predictive model with parameters to be estimated
  - A criterion to judge the goodness of fit of the model to the observations for any combination of parameter values. The criterion is often called an objective function.

$$Y_i = \hat{Y}_i + \varepsilon_i$$

    - Y_i value of observation i
    - $\hat{Y}_i$ predicted value of observation i, "the mathematical model"
    - ε_i the residual of observation i, this value is used as to calculate some criterion to judge the goodness of fit
- Parameter estimation:
  - Statistical analysis of observations (variables) where the parameters of a certain model are estimated such that the measurements are as close as possible to the predicted value given certain criteria for goodness of fit.

Goodness of fit

Observed value = Predicted value + residual

- Predicted value: Based on some formal mathematical model

Note synonyms:

- observed value = measurement
- predicted value = fitted value = expected value
- residual error = deviation = random error = residual = error = noise

Observed value = Predicted value + ε_i

$$Y_i = \hat{Y}_i + \varepsilon_i$$

- Each residual is added to the predicted value
- i stands here for a certain observation, i = 1,2,3, …n

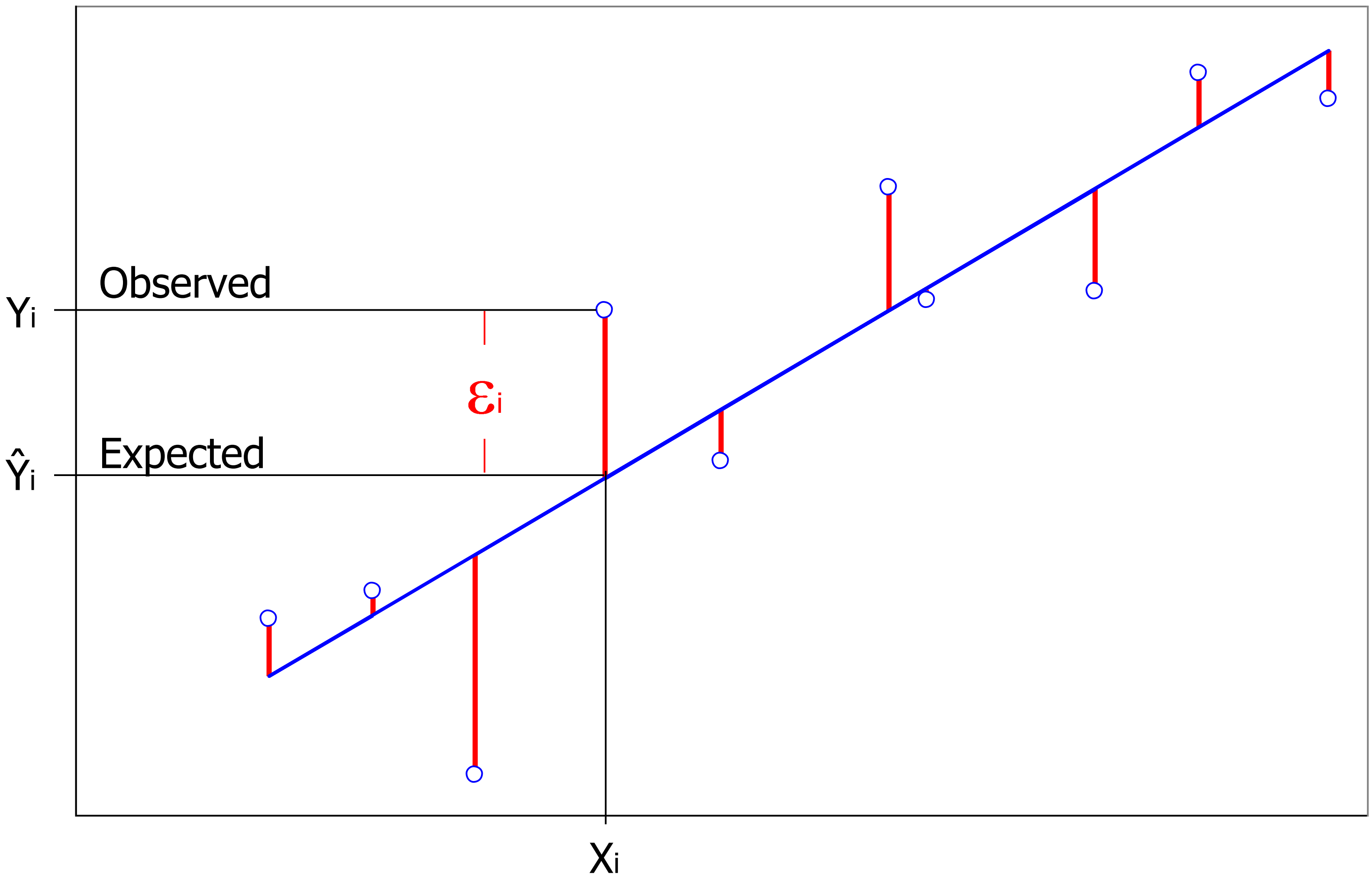
ε_i = Observed value – Predicted value

$$\varepsilon_i = Y_i - \hat{Y}_i$$

- Since the residual is a measure of distance of the prediction from that of the observed, it is an obvious candidate for measure of goodness of fit

A visual representation of residuals

$$\varepsilon_i = \hat{Y}_i - Y_i$$



Goodness of fit: Sum of squared residuals

ε_i = Observed value – Predicted value

- The deviations are both positive and negative
- We can thus not minimize the sum of the difference.

- Squaring the difference solves the problem of negative deviations

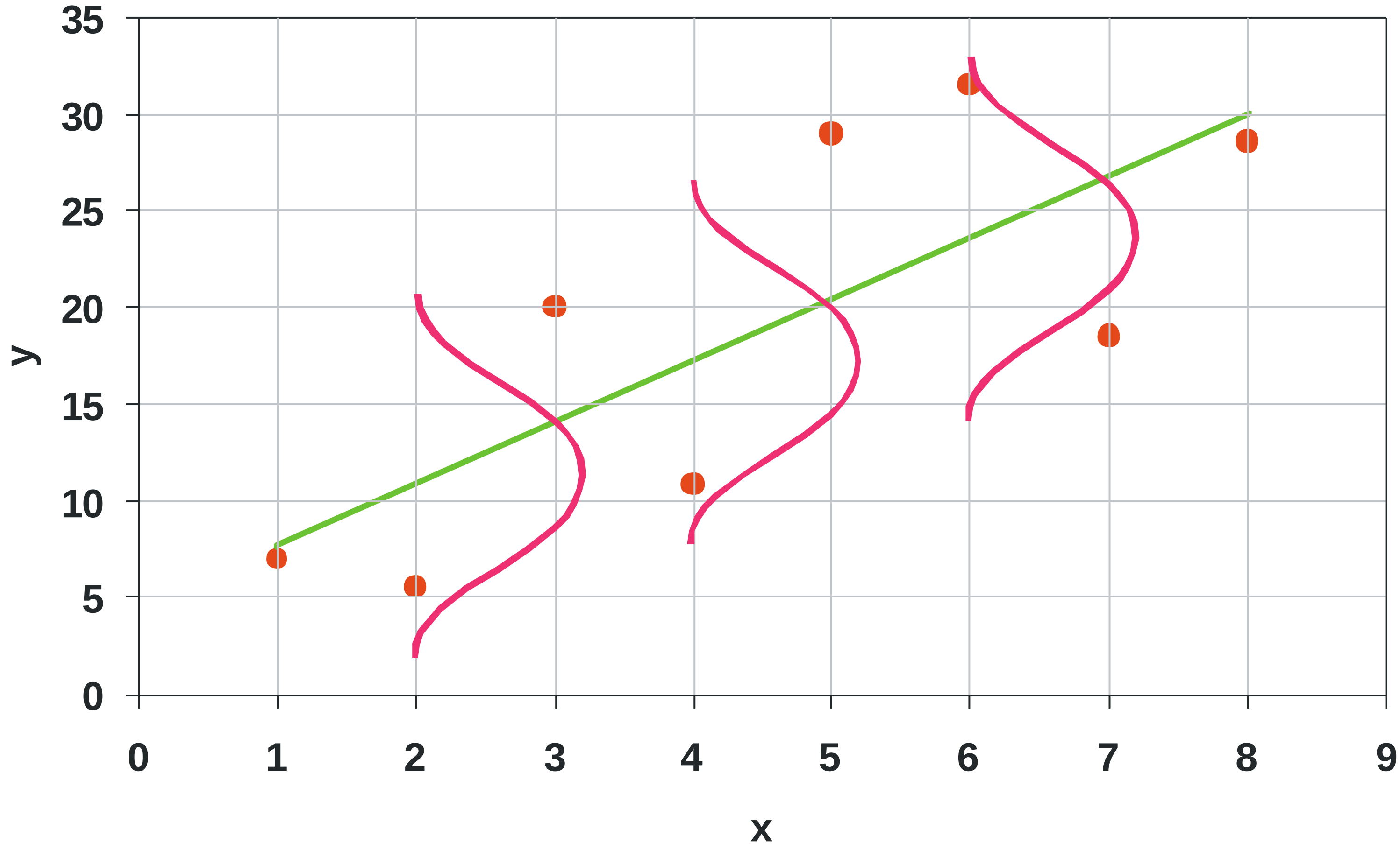
$$SS = \Sigma\varepsilon_i^2 = \Sigma(\text{Observed} - \text{Predicted})^2$$

- The criterion in the model fitting is to minimize SS

- There is are 2 major assumption when using the sums of squares as the criterion of fit; The residuals are:
  - normally distributed about the predicted variable
  - with equal variance (σ^2) for all values of the observed variable.

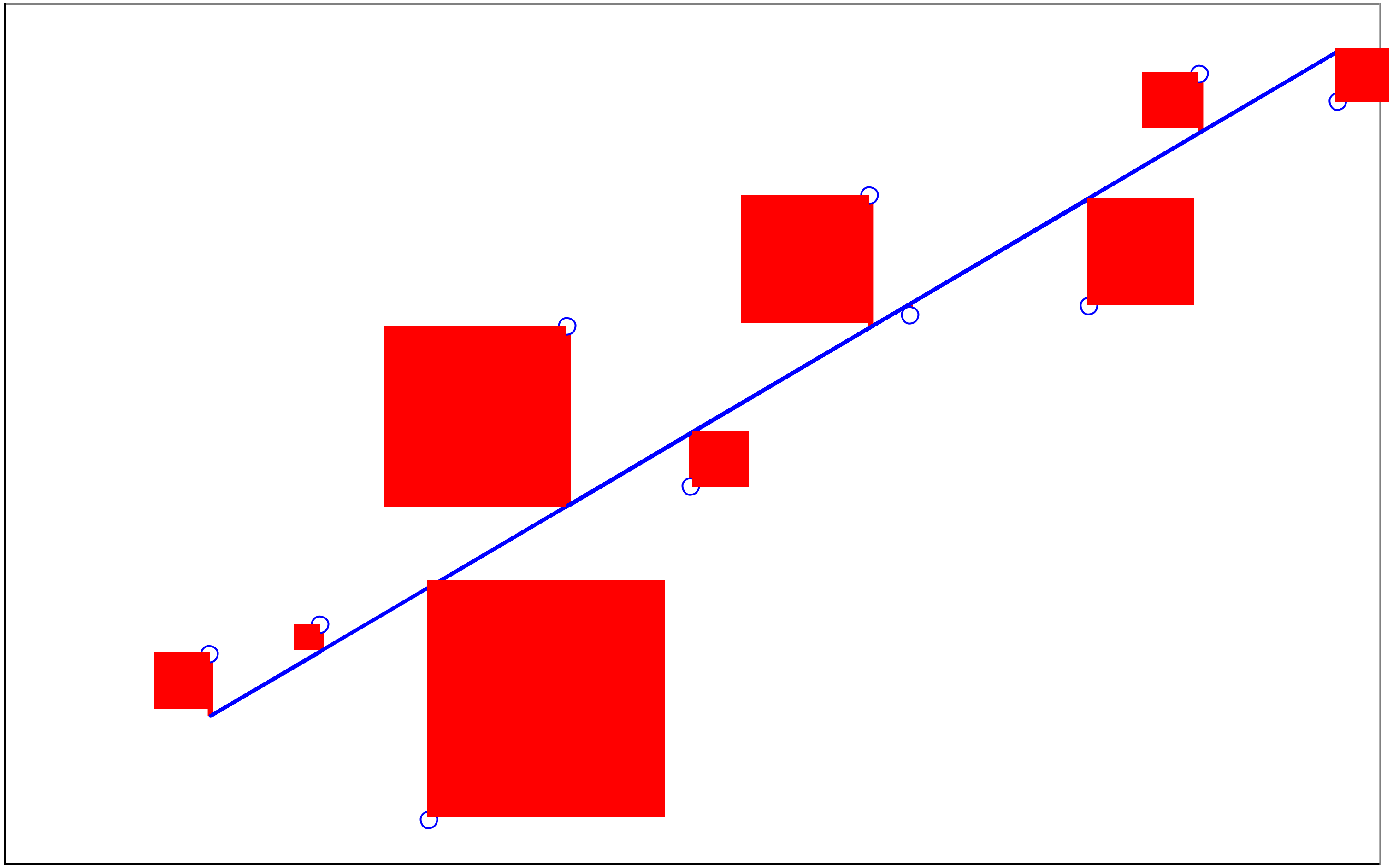
Variance of residuals independent of value of x

Often do not have sufficient data to test this assumption



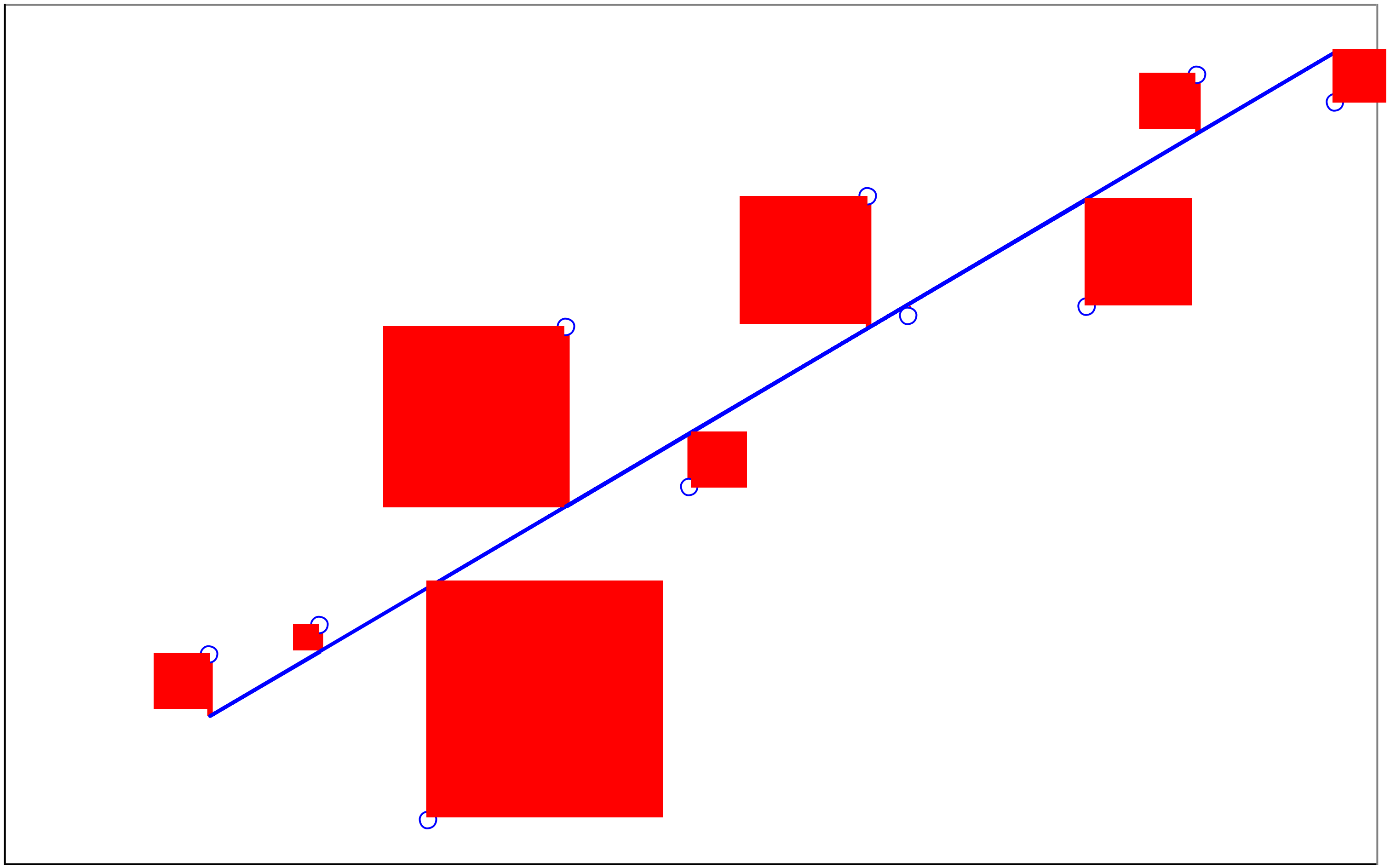
The squared residuals

$$\varepsilon_i^2 = \left(\hat{Y}_i - Y_i\right)^2$$



The sum of the squared residuals

$$\sum \varepsilon_i^2 = \sum \left(\hat{Y}_i - Y_i\right)^2 =$$ 



The principle behind the criterion

- Weather the model is simple or complex the principal of the criterion for the goodness of fit for the least square is always the same, i.e. minimize:
  - $SS = \sum (\text{Observed} - \text{Predicted})^2$
- The only complexity is the algorithm used to obtain the best parameter estimates that describe the predicted value
- With computers it easy to search numerically for values of the parameters to find the ones that fulfill the condition of minimum sums of squares
  - Grid search: Try different values for the model parameter and calculate SS for each case. The condition is still the same: Search the value for the parameters in the model that give the lowest SS.
  - Inbuilt minimization routines: Most statistical programs have these routines. They are for all practical purposes "black boxes", how it is done is not important, the principal understanding is the issue.

In Excel the black box is called Solver

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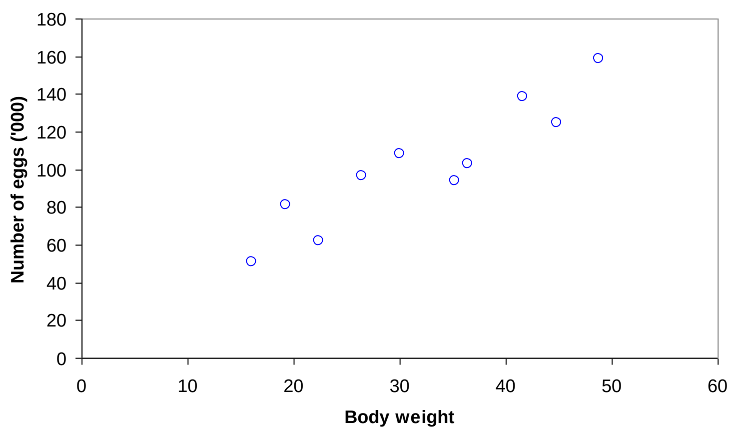
Example 1: Linear regression model with two parameters

The observations

- Ten fish (n=10) with measurements of two variables:
  - Body weight
  - Egg number

i	Weight	n Eggs
1	16.0	50.79
2	19.2	81.46
3	22.4	62.26
4	26.4	96.57
5	30.0	108.28
6	35.2	94.05
7	36.4	102.85
8	41.6	138.61
9	44.8	124.75
10	48.8	158.57

- Simple observation
  - The heavier the fish the more number of eggs
  - Proposed that a simple linear model would suffice to describe the relationship of egg number to that of fish weight.



The proposed model

- In mathematical notation we have:

  Observed = Predicted + residuals

  $Y_i = \hat{Y}_i + \varepsilon_i$

  $Y_i = a + b * X_i + \varepsilon_i$

  $\text{No. Eggs}_i = a + b * \text{Body weight}_i + \text{residual}$

- Thus for this model the goodness of fit is:

  $SS = \Sigma(\text{Observed}_i - \text{Predicted}_i)^2$

  $= \Sigma(Y_i - \hat{Y}_i)^2$

  $= \Sigma(Y_i - [a + b * X_i])^2$

  $= \Sigma(\text{No. Eggs}_i - [a + b * \text{Weight}_i])^2$

- Different values of the parameters a and b result in different values of SS. The objective is to find the combination of a and b that give the lowest SS value.
- SS: Sum of Squares

Pencil and paper calculation

Intercept (a)	20.00
Slope (b)	2.50

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.789	60.0	-9.211	84.835
2	19.2	81.462	68.0	13.462	181.214
3	22.4	62.264	76.0	-13.736	188.686
4	26.4	96.570	86.0	10.570	111.722
5	30.0	108.284	95.0	13.284	176.473
6	35.2	94.051	108.0	-13.949	194.566
7	36.4	102.849	111.0	-8.151	66.438
8	41.6	138.611	124.0	14.611	213.481
9	44.8	124.747	132.0	-7.253	52.610
10	48.8	158.565	142.0	16.565	274.407

$SS = \Sigma(Yi-\hat{Y}i)^2$ = 1544.430

Xi: Body weight

Yi: No. Eggs (´000)

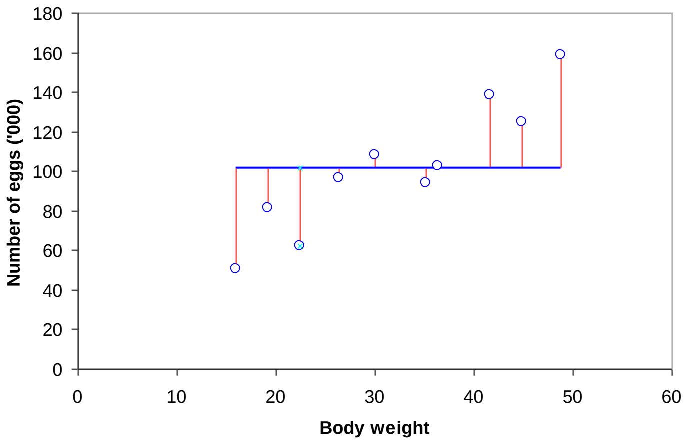
The value of a and b affect the value of SS

Intercept (a)	102.00
Slope (b)	0.00

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-Ŷi)^2$
1	16.0	50.8	102.0	-51.2	2623
2	19.2	81.5	102.0	-20.5	422
3	22.4	62.3	102.0	-39.7	1579
4	26.4	96.6	102.0	-5.4	29
5	30.0	108.3	102.0	6.3	39
6	35.2	94.1	102.0	-7.9	63
7	36.4	102.8	102.0	0.8	1
8	41.6	138.6	102.0	36.6	1340
9	44.8	124.7	102.0	22.7	517
10	48.8	158.6	102.0	56.6	3200

$SS = \Sigma(Yi-Ŷi)^2$ = 9814

Xi: Body weight
Yi: No. Eggs (´000)

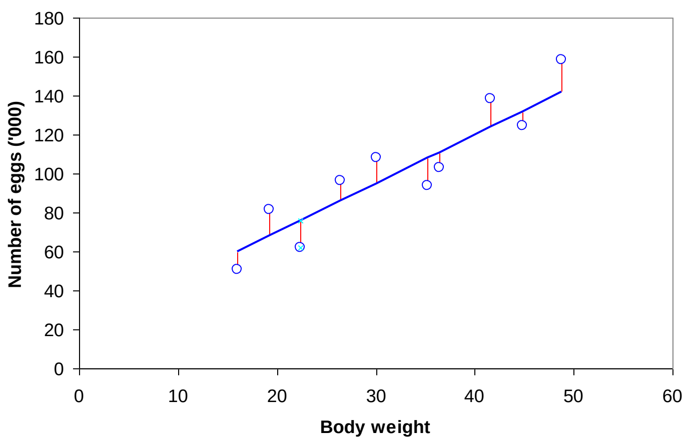


Intercept (a)	20.00
Slope (b)	2.50

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-Ŷi)^2$
1	16.0	50.8	60.0	-9.2	85
2	19.2	81.5	68.0	13.5	181
3	22.4	62.3	76.0	-13.7	189
4	26.4	96.6	86.0	10.6	112
5	30.0	108.3	95.0	13.3	176
6	35.2	94.1	108.0	-13.9	195
7	36.4	102.8	111.0	-8.2	66
8	41.6	138.6	124.0	14.6	213
9	44.8	124.7	132.0	-7.3	53
10	48.8	158.6	142.0	16.6	274

$SS = \Sigma(Yi-Ŷi)^2$ = 1544

Xi: Body weight
Yi: No. Eggs (´000)



The value of SS as a function of b

	Intercept (a)	20.00			
	Slope (b)	**2.20**			
i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.8	55.2	-4.4	19
…					
10	48.8	158.6	127.4	31.2	974

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **3042**

Yi: No. Eggs (´000)

	Intercept (a)	20.00			
	Slope (b)	**2.57**			
i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.8	61.1	-10.3	107
…					
10	48.8	158.6	145.4	13.1	173

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **1490**

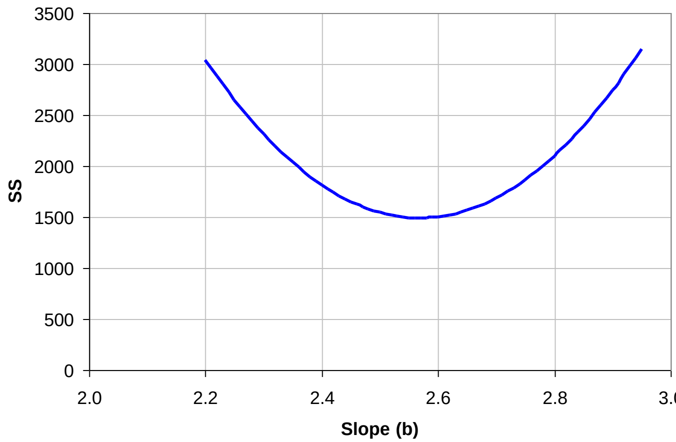
Yi: No. Eggs (´000)

	Intercept (a)	20.00			
	Slope (b)	**2.95**			
i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.789	67.2	-16.4	269
…					
10	48.8	158.565	164.0	-5.4	29

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **3148**

Yi: No. Eggs (´000)

Changing the value of the slope (b) results in a different SS value.



For a given intercept (a), there is only one value for the slope (b) that gives the lowest SS.

The value of SS as a function of a

Intercept (a)	**1.00**
Slope (b)	2.75

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.8	45.0	5.8	34
…					
10	48.8	158.6	135.2	23.4	546

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **3043**

Yi: No. Eggs (´000)

Intercept (a)	**14.00**
Slope (b)	2.75

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.8	58.0	-7.2	52
…					
10	48.8	158.6	148.2	10.4	107

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **1457**

Yi: No. Eggs (´000)

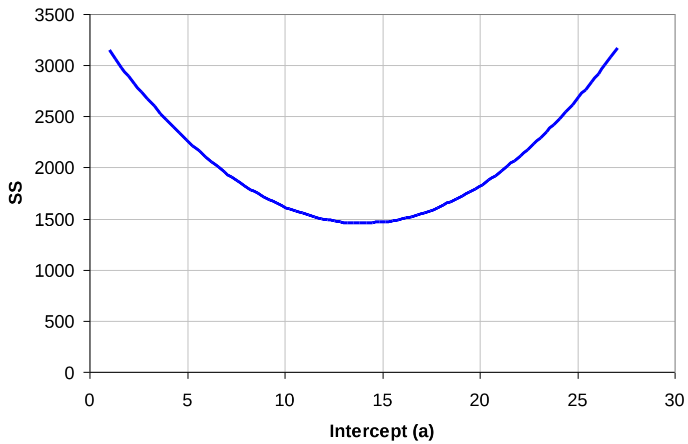
Intercept (a)	**26.00**
Slope (b)	2.75

i	Xi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	50.8	70.0	-19.2	369
…					
10	48.8	158.6	160.2	-1.6	3

Xi: Body weight — $SS = \Sigma(Yi-\hat{Y}i)^2$ **2993**

Yi: No. Eggs (´000)

Changing the value of the intercept (a) results in a different SS value.



For a given slope (b), there is only one value for the intercept (a) that gives the lowest SS.

The value of SS as a function of a and b

$$SSE = \sum_{i=1}^{n} \left(Y_i - (a + bX_i)\right)^2$$

Intercept	Slope --->									
	2.25	**2.35**	**2.45**	**2.55**	**2.65**	**2.75**	**2.85**	**2.95**	**3.05**	**3.15**
4.0	8294	6654	5243	4059	3104	2377	1878	1607	1564	1749
5.0	7791	6216	4868	3749	2858	2195	1760	1553	1574	1824
6.0	7309	5797	4514	3459	2632	2033	1662	1519	1605	1919
7.0	6846	5399	4179	3188	2426	1891	1584	1506	1655	2033
8.0	6403	5020	3865	2938	2239	**1769**	1526	1512	1726	2168
9.0	5980	4661	3571	2708	2073	1667	1489	1538	1816	2323
10.0	5577	4323	3296	2498	1927	1585	1471	1585	1927	2497
11.0	5195	4004	3042	2307	1801	1523	1473	1651	2057	2692
12.0	4832	3705	2807	2137	1695	1481	1495	1737	2208	2906
13.0	4489	3427	2593	1987	1609	1459	1537	1844	2378	3141
14.0	4166	3168	2398	1856	1543	**1457**	1599	1970	2569	3396
15.0	3864	2930	2224	1746	1496	1475	1682	2116	2779	3670
16.0	3581	2711	2069	1656	1470	1513	1784	2283	3010	3965
17.0	3318	2512	1935	1585	1464	1571	1906	2469	3260	4280
18.0	3075	2334	1820	1535	1478	1649	2048	2675	3531	4614
19.0	2852	2175	1726	1505	1512	1747	2210	2902	3821	4969
20.0	2650	2036	1651	1494	1566	1865	2393	3148	4132	5344
21.0	2467	1918	1597	1504	1640	2003	2595	3414	4462	5738
22.0	2304	1819	1562	1534	1733	2161	2817	3701	4813	6153
23.0	2161	1741	1548	1584	1847	2339	3059	4007	5183	6588

- Only one combination of a and b that gives the lowest SS.

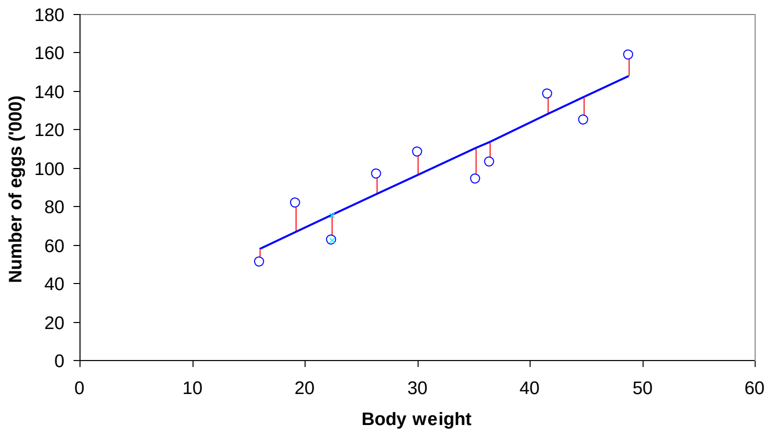
The best fit

Intercept (a)	13.99
Slope (b)	2.74

i	Xi	Yi	Ŷi	Yi-Ŷi	(Yi-Ŷi)^2
1	16.0	50.8	57.8	-7.0	49
2	19.2	81.5	66.6	14.9	222
3	22.4	62.3	75.3	-13.1	170
4	26.4	96.6	86.3	10.3	106
5	30.0	108.3	96.1	12.2	148
6	35.2	94.1	110.4	-16.3	266
7	36.4	102.8	113.6	-10.8	117
8	41.6	138.6	127.9	10.7	115
9	44.8	124.7	136.6	-11.9	142
10	48.8	158.6	147.6	11.0	120

SS = Σ(Yi-Ŷi)^2 = 1455

Xi: Body weight
Yi: No. Eggs (´000)



Lowest SS (=1455) is obtained for
Intercept = 13.99
Slope = 2.74

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Technical implementation:
Find the value of the parameters that full fills the condition of minimum sum of square of the difference

Analytical solution for linear regression

- It can be shown that the value of a and b that fulfil the condition of minimum sums of squares is the following:

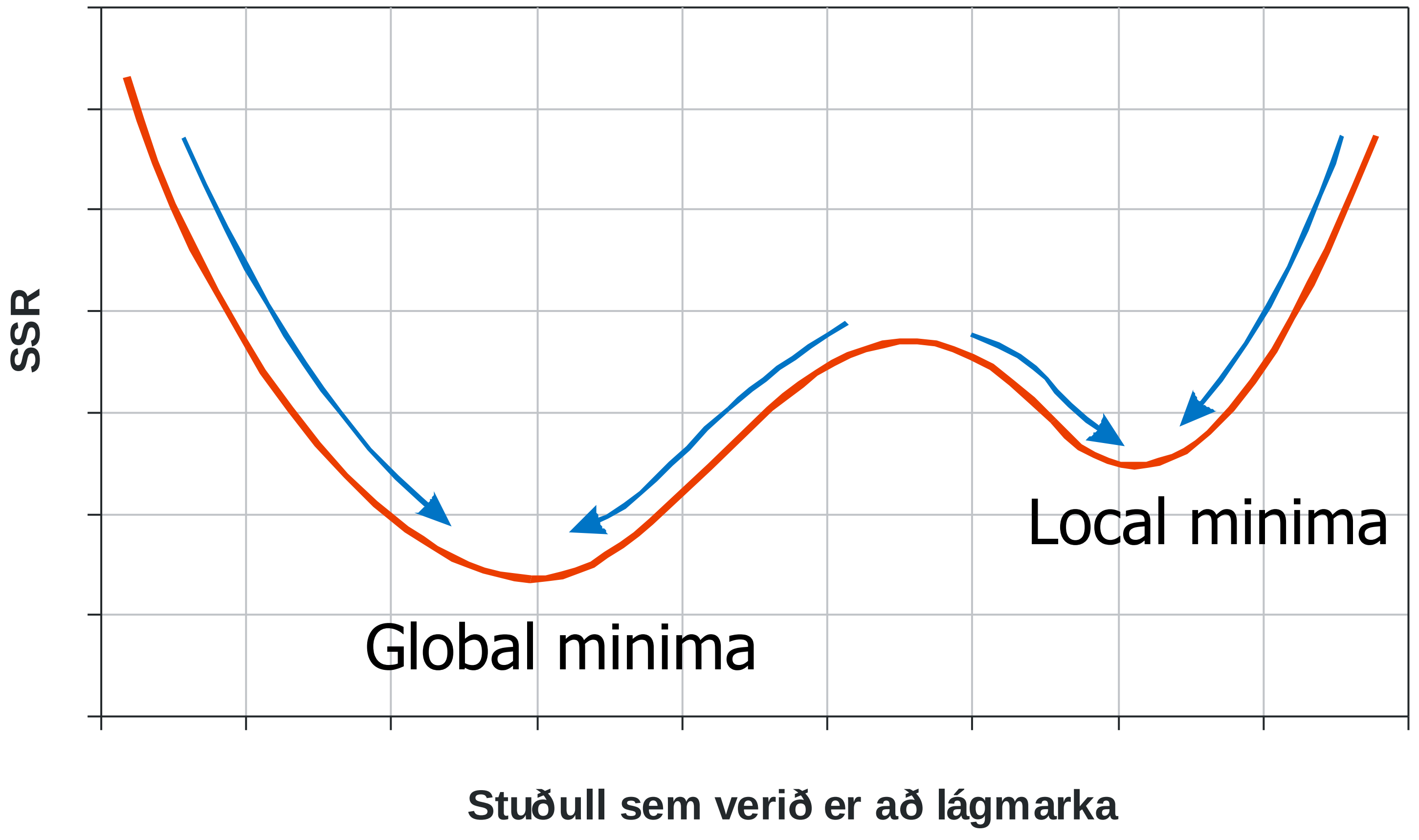
$$b = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \quad \text{og} \quad a = \bar{Y} - b\bar{X}$$

- Although an analytical solution for finding the value of parameters of interest, fulfilling the criterion of best fit, are available for a simple linear models such is not always the case for more complex models

Numerical search routines

- Analytical solution normally only available in the most simplest models. In other cases the only way is a numerical search for the parameter values that fulfils the condition of least sum of squares.
  - In Excel this is done by the use of Solver
  - In other programs: Similar methodology, just use a different name
- Disadvantages of numerical searches:
  - Can sometimes take some time
    - This is not really a problem today, high computer speed, efficient search routines
  - Can get false estimates due to local minima
    - Try different starting values for the parameters to see if the search gives same/different answeres

Local minima



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So how good is the goodness of fit?

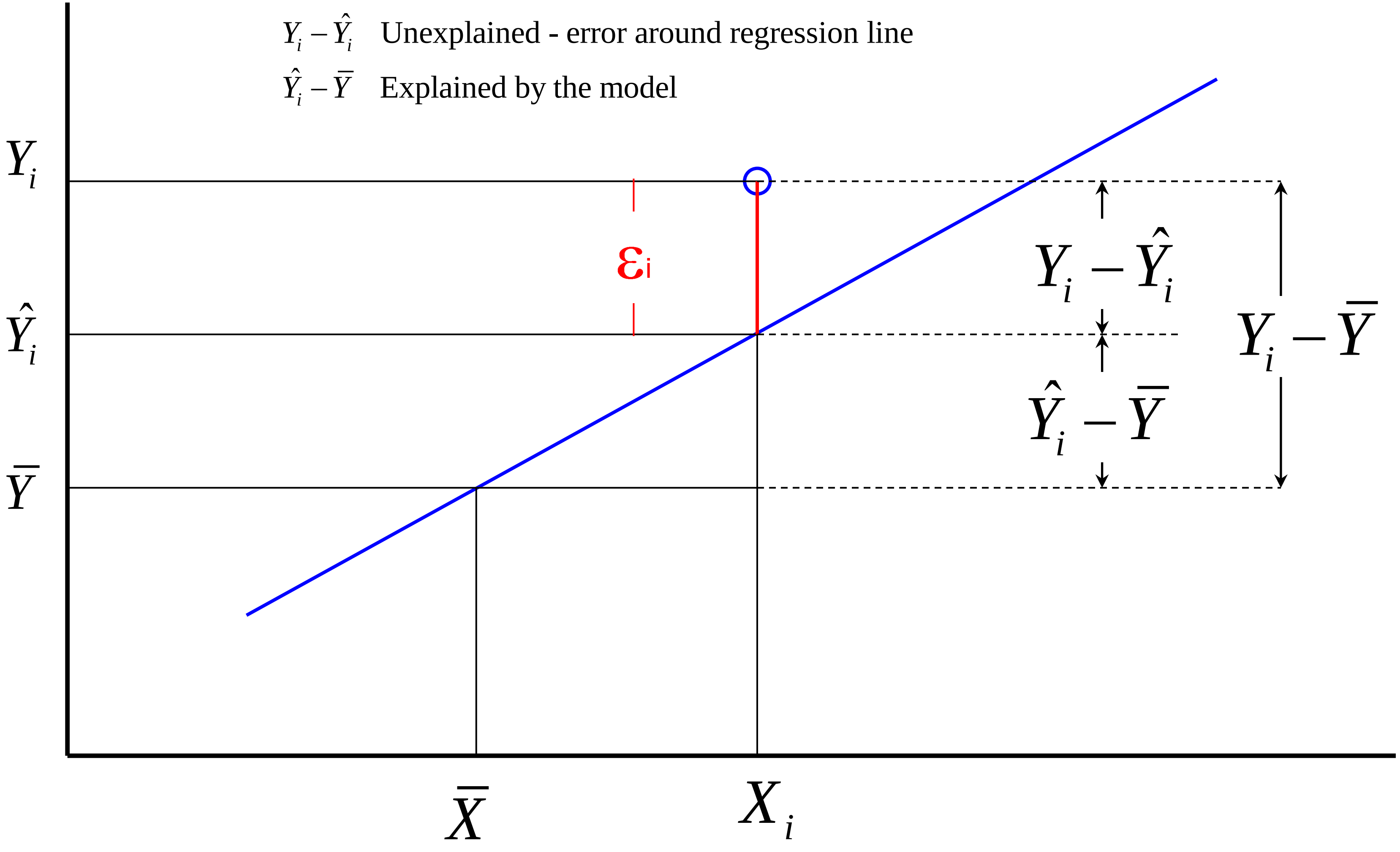
Partitioning the sums of squares

Total error = Unexplained + Explained

$Y_i - \bar{Y}$ Total error, deviation from the mean

$Y_i - \hat{Y}_i$ Unexplained - error around regression line

$\hat{Y}_i - \bar{Y}$ Explained by the model

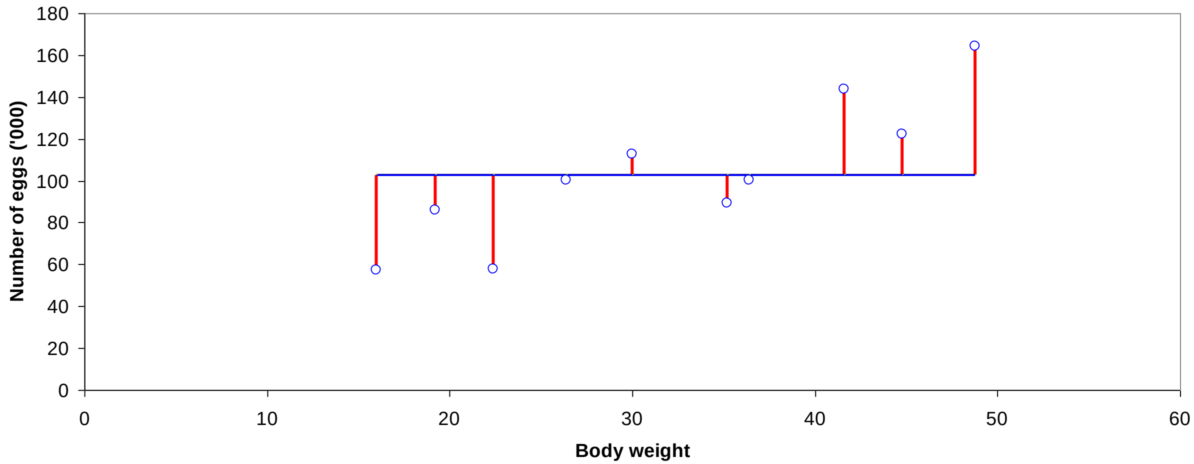


Partitioning the total variance

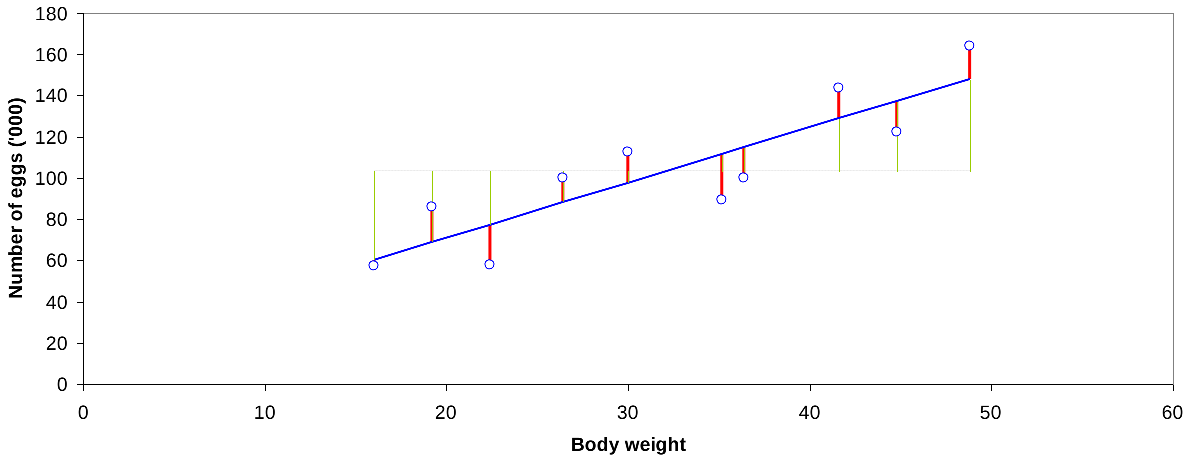
	Source of variation	df	SS	MS or S^2	
$\hat{Y}_i - \bar{Y}$	Explained	1	$SSR = \sum \left(\hat{Y}_i - \bar{Y}\right)^2$	$SSR/1$	**Variance explained by the regression**
$Y_i - \hat{Y}_i$	Unexplained	n-2	$SSE = \sum \left(Y_i - \hat{Y}_i\right)^2$	$SSE/(n-2)$	**Variance about the regression line**
$Y_i - \bar{Y}$	Total	n-1	$SST = \sum \left(Y_i - \bar{Y}_i\right)^2$	$SST/(n-1)$	**Total variance in the y-variable**

Note: Total variance = Explained variance + Unexplained variance

Partitioning the total variance



Total variance



Red: Unexplained
Green: Explained by the linear model

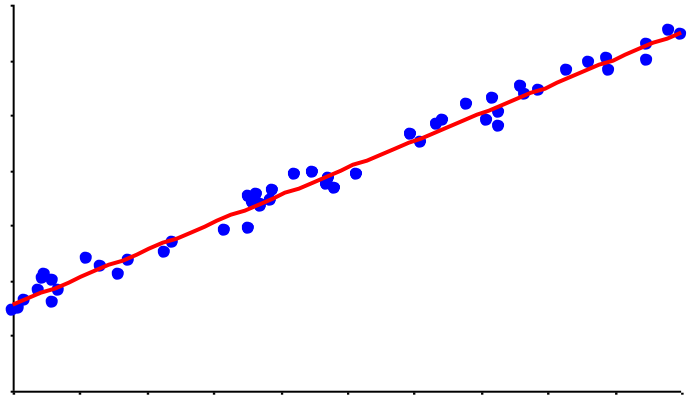
r^2 - Coefficient of determination

- Coefficient of determination is the fraction of the total variance of the dependent variable that is explained by the model.

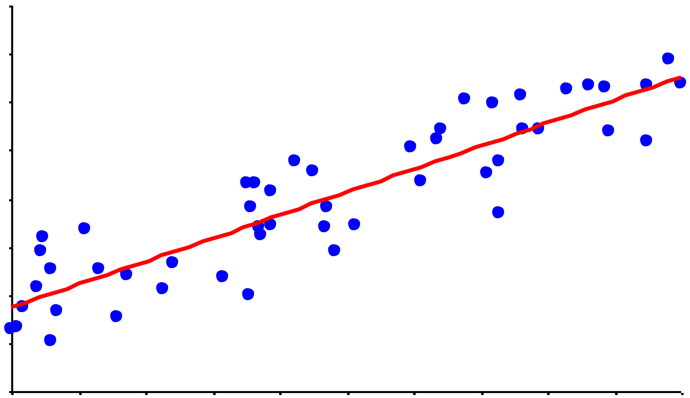
$$r^2 = \frac{\text{Explained variation}}{\text{Total variation}} = \frac{SSR}{SST} = \frac{\sum \left(\hat{Y}_i - \bar{Y}\right)^2}{\sum \left(Y_i - \bar{Y}\right)^2}$$

- The value of r^2 is between 0 and 1
- If the explained part is low compared with the total variance then r^2 will be close to 0
- If the explained part is a high proportion of the total variance then r^2 will approach 1
- Note:
- Other statistics of interest, e.g. confidence limits of the slope see statistical textbooks

Goodness of fit, standard error and r^2



- Standard deviation is low
- The unexplained variance is low and r^2 is close to 1



- Standard deviation is high
- The unexplained variance is high and r^2 is not so close to 1

Multiple regression

- Instead of having only one variable x, explaining the observations y, we may have two variables, x and z. A possible model may be:

$$y_i = a + bx_i + cz_i + \varepsilon_i$$

- The goodness of fit for this model is:

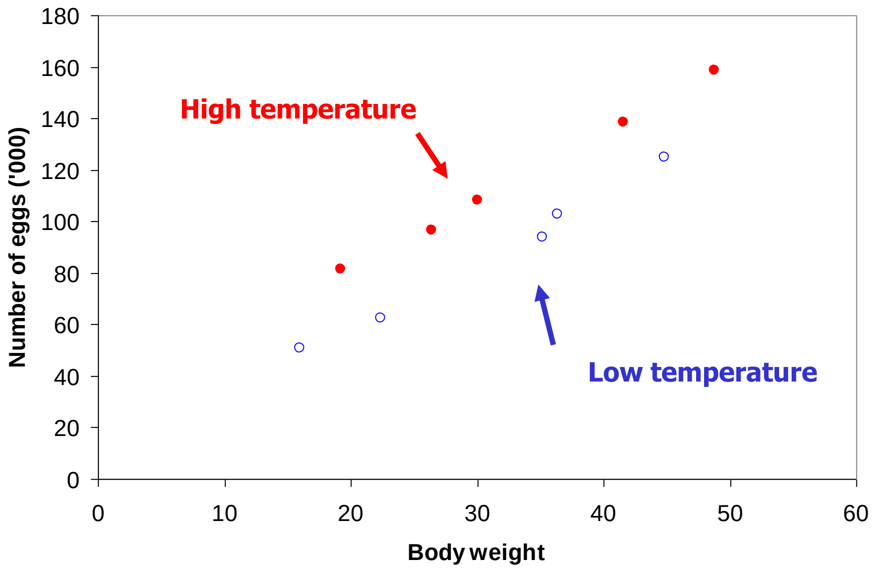
$$SS = \Sigma(\text{Observed}_i - \text{Predicted}_i)^2$$
$$= \Sigma(Y_i - \hat{Y}_i)^2$$
$$= \Sigma(Y_i - [a + b*X_i + c*Z_i])^2$$

- Different values of the parameters a, b and c result in different values of SS. The objective to find the combination of a, b and c that give the lowest SS value.

The first example extended

- Lets imagine that in our first example we also had information on temperature and that there were only two temperature regimes, 20 and 25 ºC
- Ten fish (n=10) with measurements of three variables:
  - Body weight
  - Temperature
  - Egg number
- Simple observation:
  - The heavier the fish the more number of eggs
  - For a given body weight, fish collected at lower temperature have lower number of eggs

i	Weight	Temp.	n Eggs
1	16.0	20	50.79
2	19.2	25	81.46
3	22.4	20	62.26
4	26.4	25	96.57
5	30.0	25	108.28
6	35.2	20	94.05
7	36.4	20	102.85
8	41.6	25	138.61
9	44.8	20	124.75
10	48.8	25	158.57



The proposed model

- In mathematical notation we have:

  Observed = Predicted + residuals

  $Y_i = \hat{Y}_i + \varepsilon_i$

  $Y_i = a + b * X_i + c\, Z_i + \varepsilon_i$

  No. $Eggs_i$ = a + b * Body $weight_i$ + Temperature + residual

- Thus for this model the goodness of fit is:

  $SS = \Sigma\,(Observed_i - Predicted_i)^2$

  $= \Sigma\,(Y_i - \hat{Y}_i)^2$

  $= \Sigma\,(Y_i - [a + b * X + c * Z_i])^2$

  $= \Sigma\,(No.\ Eggs_i - [a + b * Weight_i + c * Temp_i])^2$

- Different values of the parameters a, b and c result in different values of SS. The objective to find the combination of a, b and c that give the lowest SS value.
- SS: Sum of Squares

Results from the model

$$y_i = a + bx_i + cz_i + \varepsilon_i$$

Parameter	Value
Intercept (a)	-89.7
Slope (b)	2.62
Slope (c)	4.78

i	Xi	Zi	Yi	Ŷi	Yi-Ŷi	$(Yi-\hat{Y}i)^2$
1	16.0	20.0	50.8	47.8	3.0	9.1
2	19.2	25.0	81.5	80.1	1.4	2.0
3	22.4	20.0	62.3	64.5	-2.3	5.1
4	26.4	25.0	96.6	98.9	-2.3	5.4
5	30.0	25.0	108.3	108.3	-0.0	0.0
6	35.2	20.0	94.1	98.0	-4.0	15.9
7	36.4	20.0	102.8	101.2	1.7	2.8
8	41.6	25.0	138.6	138.7	-0.1	0.0
9	44.8	20.0	124.7	123.2	1.6	2.5
10	48.8	25.0	158.6	157.5	1.0	1.1

$SS = \Sigma(Yi-\hat{Y}i)^2$ = 44

Xi: Body weight
Yi: No. Eggs (´000)
Zi: Temperature

What changed?

Simple model $y_i = a + bx_i + \varepsilon_i$

Parameter	Value
Intercept (a)	14.0
Slope (b)	2.74
Slope (c)	0.00

i	Xi	Zi	Yi	Ŷi	Yi-Ŷi	$(Yi-Ŷi)^2$
1	16.0	20.0	50.8	57.8	-7.0	49.1
2	19.2	25.0	81.5	66.6	14.9	222.1
3	22.4	20.0	62.3	75.3	-13.1	170.4
4	26.4	25.0	96.6	86.3	10.3	106.1
5	30.0	25.0	108.3	96.1	12.2	147.9
6	35.2	20.0	94.1	110.4	-16.3	266.0
7	36.4	20.0	102.8	113.6	-10.8	116.6
8	41.6	25.0	138.6	127.9	10.7	115.1
9	44.8	20.0	124.7	136.6	-11.9	141.5
10	48.8	25.0	158.6	147.6	11.0	120.4

$SS = \Sigma(Yi-Ŷi)^2$ 1455

Xi: Body weight
Yi: No. Eggs (´000)
Zi: Temperature

More complex model $y_i = a + bx_i + cz_i + \varepsilon_i$

Parameter	Value
Intercept (a)	-89.7
Slope (b)	2.62
Slope (c)	4.78

i	Xi	Zi	Yi	Ŷi	Yi-Ŷi	$(Yi-Ŷi)^2$
1	16.0	20.0	50.8	47.8	3.0	9.1
2	19.2	25.0	81.5	80.1	1.4	2.0
3	22.4	20.0	62.3	64.5	-2.3	5.1
4	26.4	25.0	96.6	98.9	-2.3	5.4
5	30.0	25.0	108.3	108.3	-0.0	0.0
6	35.2	20.0	94.1	98.0	-4.0	15.9
7	36.4	20.0	102.8	101.2	1.7	2.8
8	41.6	25.0	138.6	138.7	-0.1	0.0
9	44.8	20.0	124.7	123.2	1.6	2.5
10	48.8	25.0	158.6	157.5	1.0	1.1

$SS = \Sigma(Yi-Ŷi)^2$ 44

Xi: Body weight
Yi: No. Eggs (´000)
Zi: Temperature

The simple model is a special case of the more complex model, the former being a case where we are effectively assuming that c is zero.

Note that the slopes b (the weight parameter) has not changed that much, but the intercept has (why?).

The biggest change is a much lower SS in the more complex model. The question is if the more complex model is an improvement over the simpler model??

Is the added parameter significant?

- Full (more complex) model:

$$SS_F = \sum_{i=1}^{n} \left(y_i - \left(a + bx_i + cz_i \right) \right)^2$$

- Reduced (simpler) model:

$$SS_R = \sum_{i=1}^{n} \left(y_i - \left(a + bx_i \right) \right)^2$$

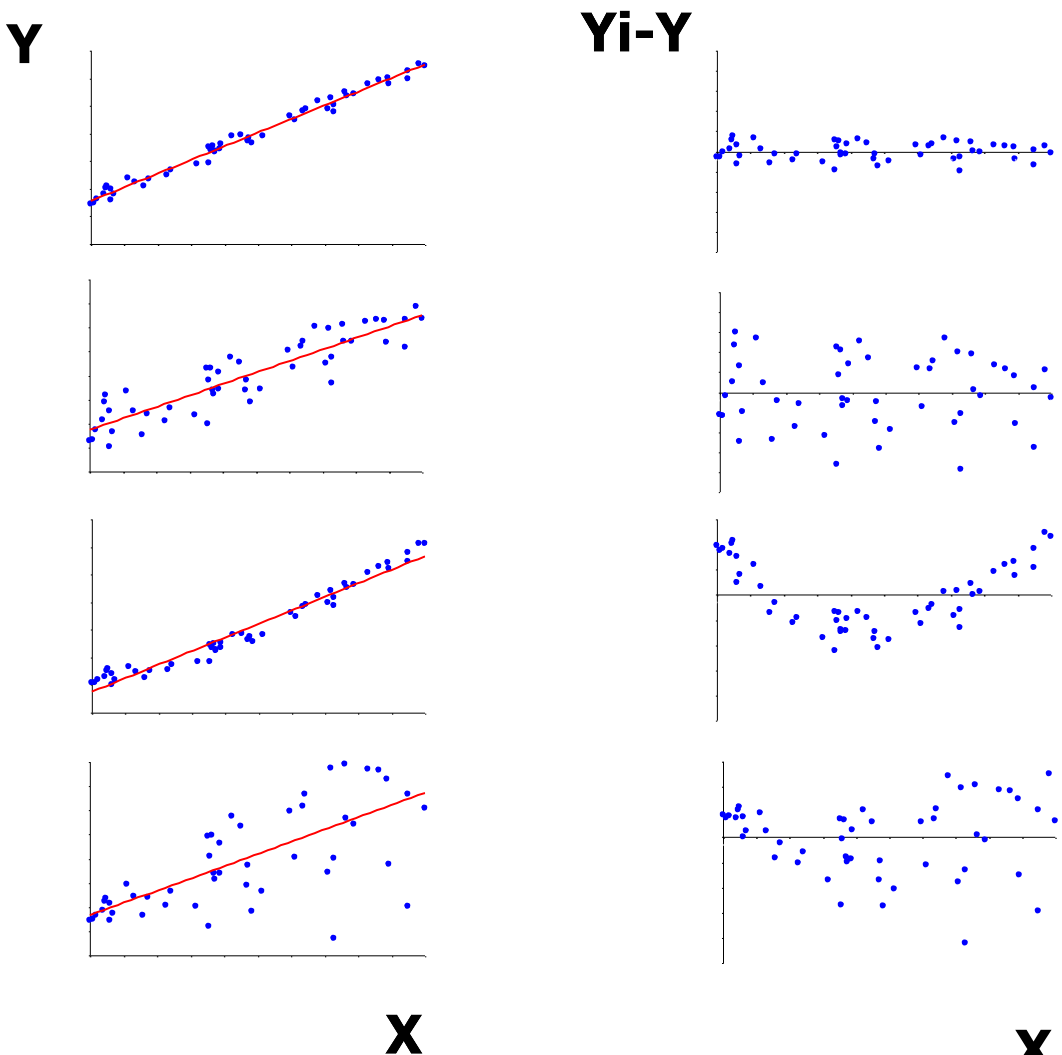
- Note that we always have: $SS_R \geq SS_F$
- if $SS_F \approx SS_S$ we have not managed to reduce the unexplained variance around the regression line.
- Formal test

$$F = \frac{\dfrac{SS_R - SS_F}{df_R - df_F}}{SS_F / df_F}$$

If F high, reject the reduced model, i.e. c is significantly different from zero.

Model violations, model transformations

Model violations: Residuals vs. independent value



OK, residuals random

OK, residuals random

Problem, residuals a function of x

Problem, variance increases with x

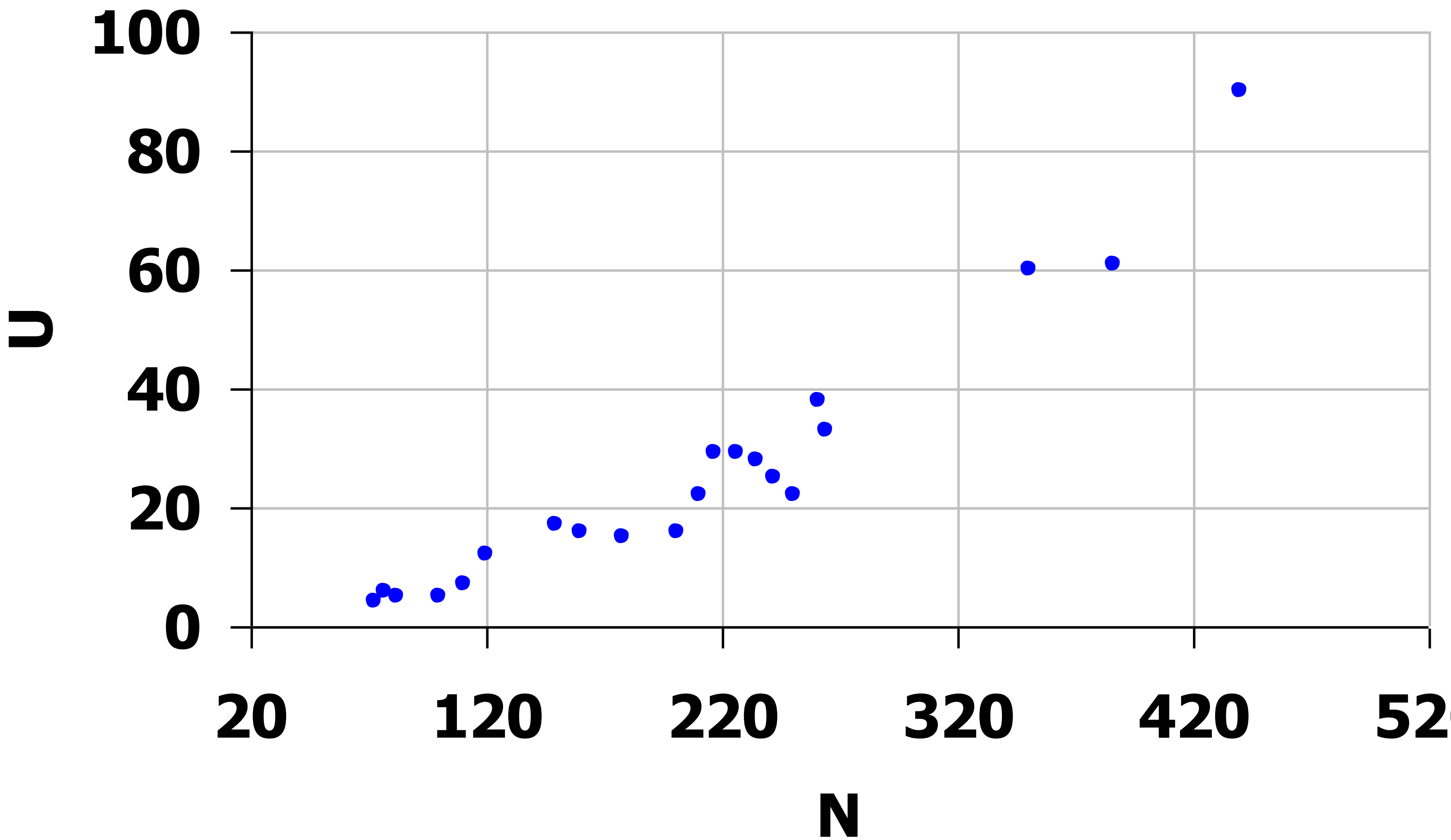
Beyond simple statistical models

- If model assumptions appear to be violated there are sometimes remedies:
  - Try alternative model if there is a systematic pattern in the residuals (often add more parameters)
  - Transformation of data if constant variance assumption is violated
  - Alternative formulation of the objective function, i.e. use some other criterion than the minimum sums of squares

Example of alternative model fitting

The measurements

N: Population numbers
U: Survey index



Lets check three alternative models

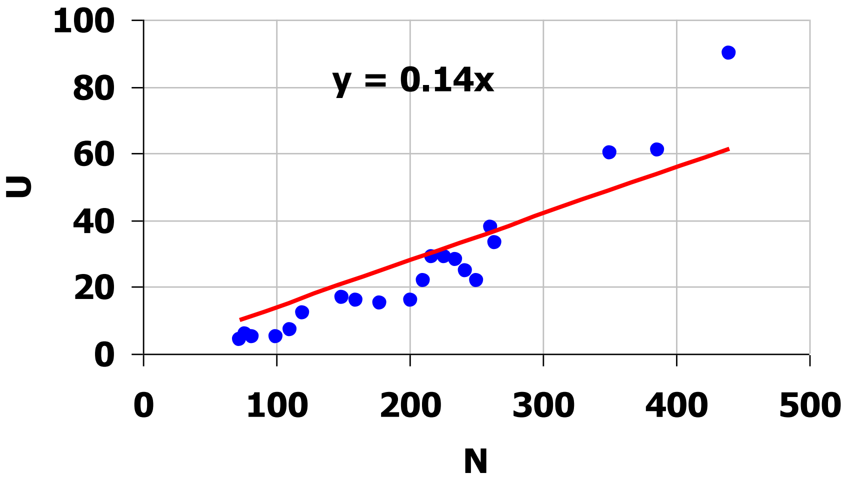
$$\hat{U}_i = bN_i, \ \mathrm{U}_\mathrm{i} = \hat{U}_i + \varepsilon_i$$

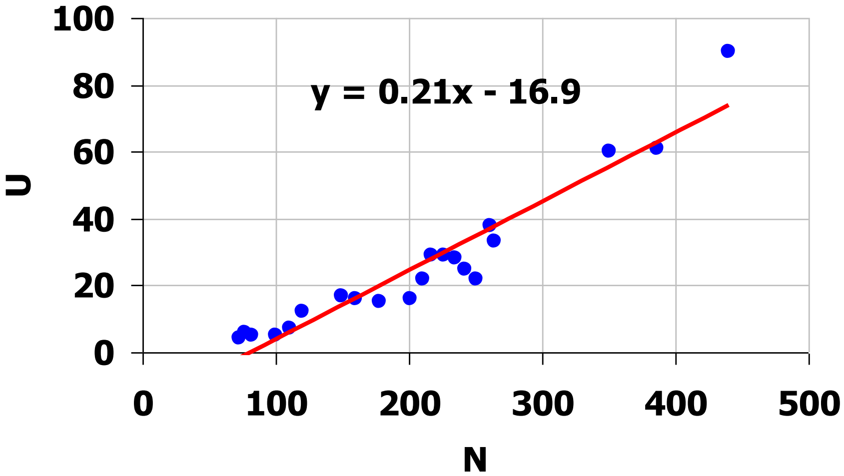
$$\hat{U}_i = a + bN_i, \ \mathrm{U}_\mathrm{i} = \hat{U}_i + \varepsilon_i$$

$$\hat{U}_i = qN_i^\beta, \ \mathrm{U}_\mathrm{i} = \hat{U}_i + \varepsilon_i$$

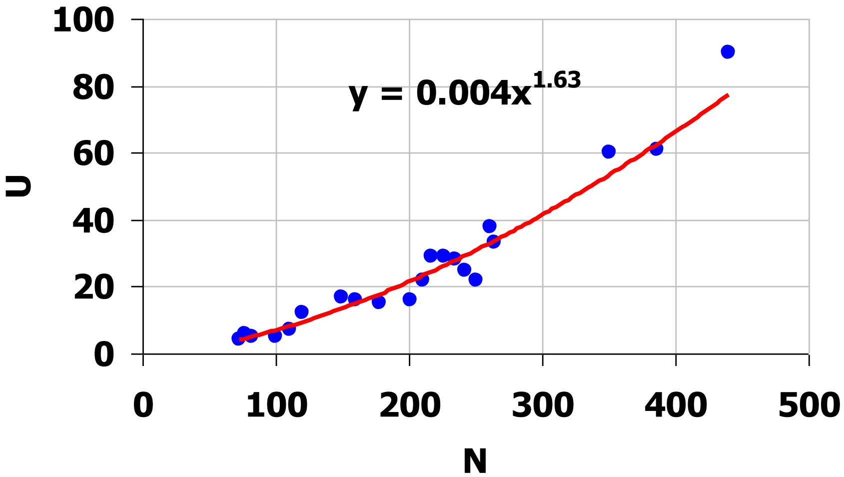
$$SSE = \sum^n \varepsilon_i^{\,2} = \sum^n \left(U_i - \hat{U}_i \right)^2$$

Models confronted with data

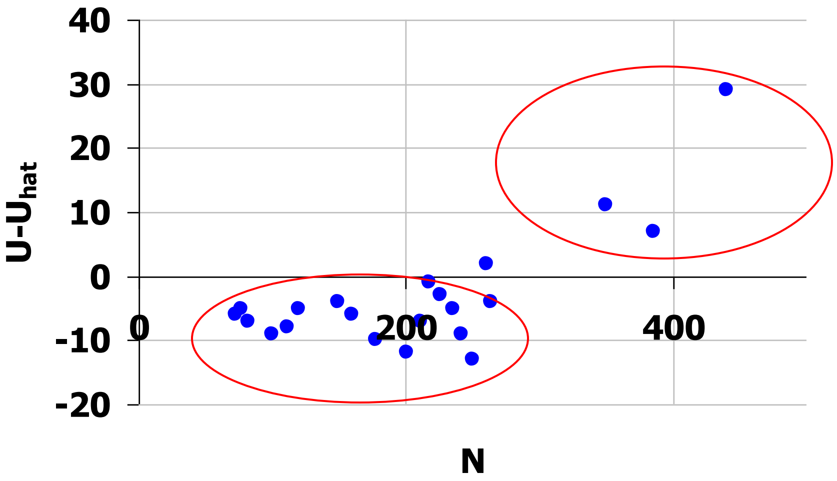


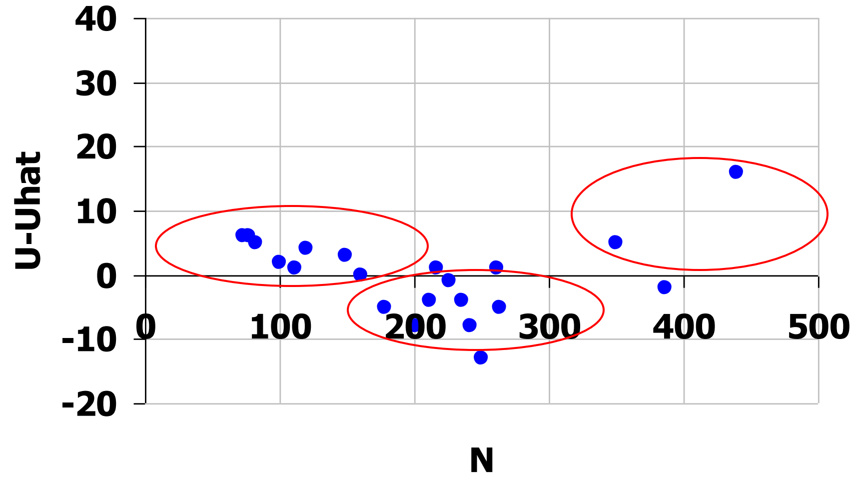


Which model has the least detectable residual problems?

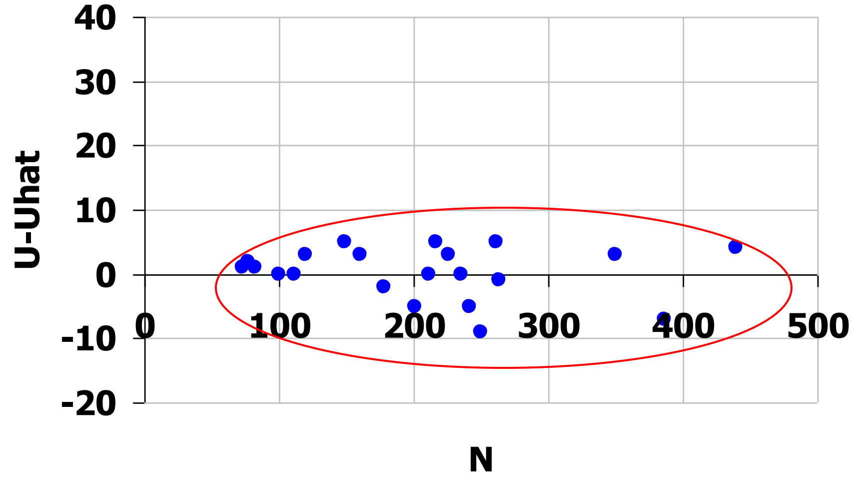


Residuals as a function of N





Power model is tends to have the "best" residual scatter pattern



Suggested further reading

- Haddon, M. 2001. Modelling and quantitative methods in fisheries.